Fidelity and information in the quantum teleportation of continuous variables

Holger F. Hofmann, Toshiki Ide, and Takayoshi Kobayashi
Department of Physics, Faculty of Science, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

Akira Furusawa
Nikon Corporation, Research and Development Headquarters, Nishi-Ohi, Shinagawa-ku, Tokyo 140-8601, Japan
(Received 16 March 2000; revised manuscript received 27 July 2000; published 9 November 2000)

Ideally, quantum teleportation should transfer a quantum state without distortion and without providing any information about that state. However, quantum teleportation of continuous electromagnetic-field variables introduces additional noise, limiting the fidelity of the quantum-state transfer. In this article, the operator describing the quantum-state transfer is derived. The transfer operator modifies the probability amplitudes of the quantum state in a shifted photon-number basis by enhancing low photon numbers and suppressing high photon numbers. This modification of the statistical weight corresponds to a measurement of finite resolution performed on the original quantum state. The limited fidelity of quantum teleportation is thus shown to be a direct consequence of the information obtained in the measurement.

PACS number(s): 03.67.Hk, 42.50.−p

I. INTRODUCTION

Quantum teleportation is a process by which the quantum state of a system A can be transferred to a remote system B by exploiting the entanglement between system B and a reference system R. Ideally, no information is obtained about system A, even though the exact relationship between A and R is determined by measuring a set of joint properties of A and R. While the original state of A is lost in this measurement, it can be recovered by deducing the relationship between A and B from the original entanglement between B and R and the measured entanglement between A and B.

The original proposal of quantum teleportation [1] assumed maximal entanglement between B and R. However, it is also possible to realize quantum teleportation with nonmaximal entanglement. In particular, such a teleportation scheme has been applied to the quantum states of light field modes [2,3], where maximal entanglement is impossible since it would require infinite energy. A schematic setup of this scheme is shown in Fig. 1. This approach to quantum teleportation has inspired a number of investigations into the dependence of the teleportation process on the details of the physical setup [4–6]. In this context, it is desirable to develop compact theoretical formulations describing the effects of this teleportation scheme on the transferred quantum state.

Originally, the teleportation process for continuous variables has been formulated in terms of Wigner functions [2]. Recently, a description in the discrete photon-number base has been provided as well [7]. In the following, the latter approach will be reformulated using the concept of displaced photon-number states, and a general transfer operator $\hat{T}(x_-,y_+)$ will be derived for the quantum teleportation of a state associated with a measurement result of x_- and y_+. This transfer operator describes the modifications that the quantum state suffers in the teleportation as well as the information obtained about the quantum state due to the finite entanglement. It is shown that this type of quantum teleportation resembles a nondestructive measurement of light field coherence with a measurement resolution given by the entanglement of B and R.

II. MEASURING THE ENTANGLEMENT OF UNRELATED FIELD MODES

The initial step in quantum teleportation requires a measurement of the entanglement between input system A and reference system R. Ideally, this projective measurement does not provide any information about properties of A by itself.

In the case of continuous field variables [2,3], the measured variables are the difference $\hat{x}_-=\hat{x}_A-\hat{x}_R$ and the sum $\hat{y}_+=\hat{y}_A+\hat{y}_R$ of the orthogonal quadrature components. The eigenstates of these two commuting variables may be expressed in terms of the photon-number states $|n_A;n_R\rangle$ as

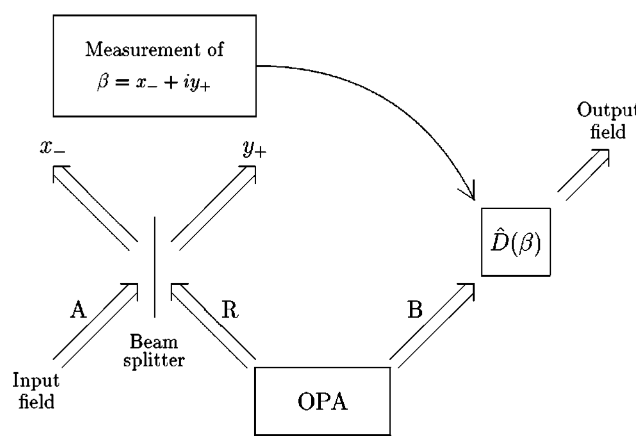

FIG. 1. Schematic representation of the quantum teleportation scheme.

$$|\beta(A,R)\rangle = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\hat{D}_A(\beta)|n;n\rangle$$
$$\text{with}\quad \hat{x}_-|\beta(A,R)\rangle = \mathrm{Re}(\beta)|\beta(A,R)\rangle$$
$$\text{and}\quad \hat{y}_+|\beta(A,R)\rangle = \mathrm{Im}(\beta)|\beta(A,R)\rangle, \tag{1}$$

where the operator $\hat{D}_A(\beta)$ is the displacement operator acting on the input field A, such that

$$\hat{D}_A(\beta) = \exp(2i\,\mathrm{Im}(\beta)\hat{x}_A - 2i\,\mathrm{Re}(\beta)\hat{y}_A)$$
$$\text{with}\quad \hat{D}_A^\dagger(\beta)\hat{x}_A\hat{D}_A(\beta) = \hat{x}_A + \mathrm{Re}(\beta)$$
$$\text{and}\quad \hat{D}_A^\dagger(\beta)\hat{y}_A\hat{D}_A(\beta) = \hat{y}_A + \mathrm{Im}(\beta). \tag{2}$$

Of course, the coherent shift could also be applied to field R instead of field A. However, in the representation given by Eq. (1), it is easy to identify the quantum state associated with a photon number of the reference field R.

If the quantum state $|\psi_R\rangle$ of the reference field R is known, the measurement result provides information on the quantum state $|\psi_A\rangle$ of field A through the probability distribution $P(\beta)$ given by

$$P(\beta) = \frac{1}{\pi}\left|\sum_{n=0}^{\infty}\langle\psi_A|\hat{D}_A(\beta)|n\rangle\langle\psi_R|n\rangle\right|^2$$
$$= \frac{1}{\pi}|\langle\psi_A|\hat{D}_A(\beta)|\psi_R^*\rangle|^2,$$
$$\text{where}\quad |\psi_R^*\rangle = \sum_{n=0}^{\infty}\langle n|\psi_R\rangle^*|n\rangle. \tag{3}$$

Effectively, the measurement of entanglement projects the quantum state of field A onto a complete nonorthogonal measurement basis given by the displaced reference states $|\psi_R^*\rangle$. The completeness of this measurement basis is given by

$$\frac{1}{\pi}\int d^2\beta\,\hat{D}(\beta)|\psi_R^*\rangle\langle\psi_R^*|\hat{D}^\dagger(\beta) = \hat{1}. \tag{4}$$

In the case of ''classical'' teleportation, the reference field R is in the quantum-mechanical vacuum state $|n=0\rangle$. Therefore, the measurement of the field entanglement given by β projects the incoming signal field A directly onto a displaced vacuum state.

III. QUANTUM TELEPORTATION

In the general case of quantum teleportation, the quantum state of the reference field R cannot be determined locally because of its entanglement with the remote field B. This indicates that the type of measurement performed is unknown until the remote system is measured as well. The meaning of the measurement result β depends on the unknown properties of the remote field B. The entangled reference field R thus provides the means to choose between complementary measurement types even after the measurement interaction between input field A and reference field R has occurred.

Within the quantum-state formalism, the initial state of the entangled fields R and B may be written as [7,8]

$$|q(R,B)\rangle = \sqrt{1-q^2}\sum_{n=0}^{\infty}q^n|n;n\rangle. \tag{5}$$

Thus, the photon numbers of the reference field R and the remote field B are always equal. However, low photon numbers are more likely than high photon numbers, so the two-mode entanglement is limited by the information available about the photon number of each mode. In a measurement of the entanglement between field A and field R, this information about R is converted into measurement information about A, thus causing a decrease in fidelity as required by the uncertainty principle.

A measurement of the entanglement between field A and field R projects the product state $|\psi_A\rangle\otimes|q(R,B)\rangle$ into a quantum state of the remote field B given by

$$|\psi_B(\beta)\rangle = \sqrt{\frac{1-q^2}{\pi}}\sum_{n=0}^{\infty}q^n|n\rangle\langle n|\hat{D}_A(-\beta)|\psi_A\rangle, \tag{6}$$

where the measurement probability $P(\beta)$ is given by $\langle\psi_B(\beta)|\psi_B(\beta)\rangle$. Thus the measurement determines the displacement β between field A and field B, resulting in a quantum state $|\psi_B(\beta)\rangle$ that appears to be a copy of the input state $|\psi_A\rangle$, displaced by $-\beta$. However, the measurement information obtained because low photon numbers are more likely than high photon numbers in both R and B causes a statistical modification of the probability amplitudes of the photon-number states in the remote field B.

The final step in quantum teleportation is the reconstruction of the initial state from the remote field by reversal of the displacement. The output state then reads

$$|\psi_{\mathrm{out}}(\beta)\rangle = \hat{T}(\beta)|\psi_A\rangle$$
$$\text{with}\quad \hat{T}(\beta) = \sqrt{\frac{1-q^2}{\pi}}\sum_{n=0}^{\infty}q^n\hat{D}_A(\beta)|n\rangle\langle n|\hat{D}_A(-\beta). \tag{7}$$

Note that this output state is not normalized because $\langle\psi_{\mathrm{out}}(\beta)|\psi_{\mathrm{out}}(\beta)\rangle$ defines the probability of the measurement results β. The complete process of quantum teleportation is thus summed up by the transfer operators $\hat{T}(\beta)$.

IV. TRANSFER OPERATOR PROPERTIES

The transfer operator $\hat{T}(\beta)$ determines not only the properties of the quantum state after the teleportation process following a measurement result of β for the field entanglement between A and R, but also the probability of obtaining the result β itself. The probability distribution $P(\beta)$ is given by

$$P(\beta)=\langle\psi_A|\hat{T}^2(\beta)|\psi_A\rangle = \frac{1-q^2}{\pi}\sum_{n=0}^{\infty} q^{2n}\langle n|\hat{D}_A(-\beta)|\psi_A\rangle|^2. \tag{8}$$

Since the prefactor q^n is larger for small n, a measurement result of β is more likely if the photon number of the displaced state $\hat{D}_A(-\beta)|\psi_A\rangle$ is low. It is possible to identify the displaced photon number with the square of the field difference between β and the actual field value of A. Therefore, a measurement result of β makes large deviations of the field A from this value of β unlikely.

The transfer operator $\hat{T}(\beta)$ also determines the relationship between the input state and the output state. Since it is the goal of quantum teleportation to achieve identity between the input state and the output state, the overlap between the two states may be used as a measure of the fidelity of quantum teleportation. For a single teleportation event associated with a measurement result of β, this fidelity is given by

$$F(\beta)=\frac{1}{P(\beta)}|\langle\psi_A|\hat{T}(\beta)|\psi_A\rangle|^2. \tag{9}$$

For instance, the fidelity of quantum teleportation for a photon-number state displaced by β is exactly 1. However, it is unlikely that β will be exactly equal to the displacement of the photon-number state to be teleported, so the average fidelity will be much lower. The average fidelity $F_{\text{av.}}$ is given by

$$F_{\text{av.}}=\int d^2\beta\, P(\beta)F(\beta)=\int d^2\beta|\langle\psi_A|\hat{T}(\beta)|\psi_A\rangle|^2. \tag{10}$$

Since the transfer operator $\hat{T}(\beta)$ is different for each teleportation event, the output field states show unpredictable fluctuations. These fluctuations may be expressed in terms of a density matrix

$$\hat{\rho}_{\text{out}}=\int d^2\beta\hat{T}(\beta)|\psi_A\rangle\langle\psi_A|\hat{T}(\beta). \tag{11}$$

In terms of this mixed-state density matrix, the average fidelity reads

$$F_{\text{av.}}=\langle\psi_A|\hat{\rho}_{\text{out}}|\psi_A\rangle. \tag{12}$$

However, the measurement information β is available as classical information, so the density matrix $\hat{\rho}_{\text{out}}$ actually underestimates the information available about the output field. In particular, a verifier checking the fidelity of the transfer in B can know the exact output state based on the knowledge of the input state and the measurement result β.

V. FIDELITY AND INFORMATION

The transfer operator $\hat{T}(\beta)$ describes how the information β obtained about the properties of the input state makes contributions from displaced photon-number states less likely as the displaced photon-number increases. The changes in the quantum state that are responsible for a fidelity below one correspond to this change in the statistical weight of the quantum-state components. This situation is typical for quantum-mechanical measurements, since maximal quantum coherence between two components of the quantum-state requires equal probability amplitudes for each component. Making one quantum-state component more likely than another necessarily diminishes quantum coherence [9].

In order to clarify the type of measurement information obtained, it is convenient to represent the transfer operator $\hat{T}(\beta)$ in terms of coherent states. This representation is easy to obtain by using the formal analogy of $\hat{T}(\beta)$ with a thermal photon-number distribution. The result reads

$$\hat{T}(\beta)=\sqrt{\frac{1-q^2}{\pi^3q^2}}\int d^2\alpha\exp\left(-\left(\frac{1-q}{q}\right)|\alpha-\beta|^2\right)|\alpha\rangle\langle\alpha|. \tag{13}$$

In the limit of $q\to 0$, the transfer operator thus corresponds to a projection operator on the coherent state $|\beta\rangle$. As q increases, the operator corresponds to a mixture of weighted projections that prefer coherent states with field values close to β, distorting the field distribution of the input state.

The information obtained in the measurement of β about the original quantum state $|\psi_A\rangle$ is therefore information about the coherent field amplitude. The loss of fidelity is a necessary consequence of the projective nature of quantum measurements [9,10]. If one keeps track of the information contained in the measurement result β and combines it with information obtained from measurements of the output state $|\psi_{\text{out}}(\beta)\rangle$, the complete process of information transfer from A to B can be described by a single quantum measurement applied directly to the original input state $|\psi_A\rangle$.

VI. VERIFICATION OF QUANTUM-STATE STATISTICS

The output of a single teleportation process involving a pure-state input $|\psi_A\rangle$ results in a well-defined pure-state output $\hat{T}(\beta)|\psi_A\rangle$. Since the statistical properties of this state are modified by the information obtained in the measurement of β, the output state is different from the input state, as given by the fidelity $F(\beta)$ defined by Eq. (9). However, this difference shows only in the statistics obtained by measuring the output of an ensemble of identical input states. In other words, there is no single measurement to tell us whether the output quantum state is actually identical to the input state. Nonorthogonal quantum states may always produce the same measurement results. The verification process following the quantum teleportation is therefore a nontrivial process requiring the comparison of measurement statistics that are generally noisy, even for a fidelity of one.

In the experimentally realized teleportation of continuous variables reported in [3], the verification is achieved by measuring one quadrature component of the light field using homodyne detection and comparing the result with the quadrature noise of the coherent-state input. This type of verification can be generalized as a projective measurement on a set of states $|V\rangle$ satisfying the completeness condition for positive-valued operator measures,

$$\sum_V |V\rangle\langle V| = \hat{1}. \tag{14}$$

The probability of obtaining a verification result V is given by

$$P(V) = \int d^2\beta |\langle V|\hat{T}(\beta)|\psi_A\rangle|^2. \tag{15}$$

This probability distribution is then compared with the input distribution of the verification variable V. However, the total process of teleportation and verification may be summarized in a single measurement of β and V. If the information inherent in the measurement result β is retained, the complete measurement performed on the input state is defined by the projective measurement basis $|\beta,V\rangle$ given by

$$|\beta,V\rangle = \hat{T}(\beta)|V\rangle. \tag{16}$$

The probability distribution over measurement results β and verification results V then reads

$$P(\beta,V) = |\langle\beta,V|\psi_A\rangle|^2 = |\langle V|\hat{T}(\beta)|\psi_A\rangle|^2. \tag{17}$$

The quantum measurement effectively performed on the input state is thus composed of the measurement step of quantum teleportation and the verification step. The fidelity is determined by the difference in the statistics over V between this two-step measurement and a direct measurement of V only. However, the information lost in quantum teleportation is actually less than is suggested by the average fidelity. If the teleportation result β is considered as well, the combination of teleportation and verification extract the maximal amount of measurement information permitted in quantum mechanics and consequently allows a complete statistical characterization of the original input state.

A particularly striking example can be obtained for a verification scheme using eight-port homodyne detection. In this case, the verification variable is the coherent field α and the verification states $|V\rangle = 1/\sqrt{\pi}|\alpha\rangle$ are the associated coherent states. The effective measurement basis is then given by

$$|\beta,\alpha\rangle = \frac{1}{\sqrt{\pi}}\hat{T}(\beta)|\alpha\rangle = \frac{\sqrt{1-q^2}}{\pi}\exp\left(-(1-q^2)\frac{|\alpha-\beta|^2}{2}\right) \times|\gamma = \beta + q(\alpha-\beta)\rangle. \tag{18}$$

The measurement still projects on a well-defined coherent state, but the coherent field γ is a function of both the teleportation measurement β and the verification measurement α. It is therefore possible to reconstruct the correct measurement statistics of the input state by referring to the teleportation results β as well as to the verification results α.

VII. CONCLUSIONS

In conclusion, the quantum teleportation of an input state $|\psi_A\rangle$ can be described by a measurement-dependent transfer operator $\hat{T}(\beta)$ that modifies the quantum-state statistics according to the information obtained about the input state $|\psi_A\rangle$ in the measurement of β. The statistics of subsequent verification measurements may be derived by directly applying the transfer operator $\hat{T}(\beta)$ to the states $|V\rangle$ describing the projective verification measurement. Quantum information is only lost because the effective measurement basis $|\beta,V\rangle$ does not usually correspond to the eigenstate basis in which the information has been encoded.

While the type of physical information that can be obtained about the original input field is restricted because some information necessarily ''leaks out'' in quantum teleportation with limited entanglement, the total information obtained after the verification step still corresponds to the information obtained in an ideal projective measurement. The limitations imposed on quantum teleportation by a fidelity less than one are thus a direct consequence of the measurement information obtained about the transferred state in the measurement of β and can be applied directly to the measurements performed after the transfer of the quantum state.

ACKNOWLEDGMENT

One of us (H.F.H.) would like to acknowledge support from the Japanese Society for the Promotion of Science, JSPS.

[1] C.H. Bennet, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W.K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[2] S.L. Braunstein and H.J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).

[3] A. Furusawa, J.L. Sorensen, S.L. Braunstein, C.A. Fuchs, H.J. Kimble, and E.S. Polzik, Science **282**, 706 (1998).

[4] G.J. Milburn and S.L. Braunstein, Phys. Rev. A **60**, 937 (1999).

[5] T. Opatrny, G. Kurizki, and D.G. Welsch, Phys. Rev. A **61**, 032302 (2000).

[6] D.B. Horoshko and S.Y. Kilin, Phys. Rev. A **61**, 032304 (2000).

[7] S.J. van Enk, Phys. Rev. A **60**, 5095 (1999).

[8] D.F. Walls and G.J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1994).

[9] H.F. Hofmann, Phys. Rev. A **62**, 022103 (2000).

[10] A discussion of the interpretational aspects of this effect is given in the e-print version of this paper, quant-ph/0003053.